The University of Chicago
FOUNDED BY JOHN D. ROCKEFELLER

Group-Characters of Various Linear Groups

A DISSERTATION

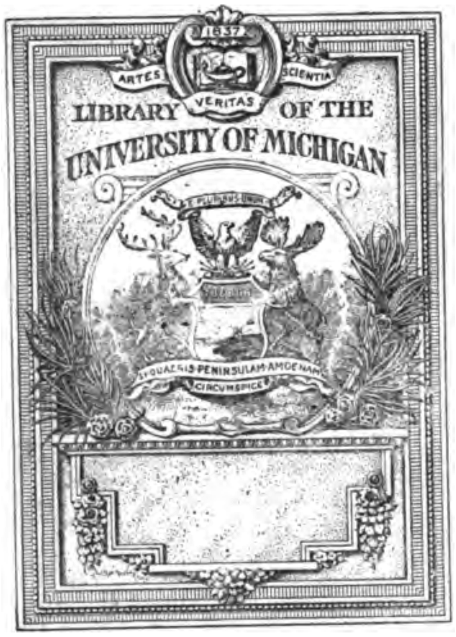
SUBMITTED TO THE FACULTY OF THE OGDEN GRADUATE SCHOOL OF SCIENCE
IN CANDIDACY FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

BY

HERBERT E. JORDAN

The Lord Baltimore Press
BALTIMORE, MD., U. S. A.
1907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*Group-Characters of Various Types of Linear Groups.**

BY HERBERT E. JORDAN.

INTRODUCTION.

In an article entitled Über Gruppencharaktere† Frobenius has determined the group-characters of the group of all binary linear fractional substitutions of determinant unity (when in their normal forms), the coefficients being taken modulo p, an odd prime. In the present paper the same method is applied to more general types of groups.

In Part I we consider the group $H \equiv SLH(2, p^n)$, $p > 2$, of all binary linear homogeneous substitutions in the $GF[p^n]$, of determinant unity. By the aid of two theorems due to Frobenius on the relation between the characters of a group and those of one of its quotient-groups, we deduce as a corollary the characters of the group $F \equiv LF(2, p^n)$, $p > 2$, of all binary linear fractional substitutions in the $GF[p^n]$ of determinant unity (when in their normal forms). We have also obtained these characters directly by the method applied to the group H; the chief points of difference in the treatment are stated in foot-notes. The results are a direct generalization of those obtained by Frobenius. In Part II we consider the group $H_1 \equiv SLH(2, p^n)$, $p = 2$. This is identical with the group $LF(2, p^n)$, $p = 2$. Part III deals with the group F_1 of all binary linear fractional substitutions in the $GF[p^n]$, $p > 2$, of determinant not zero. The group H is treated with considerable detail; the others briefly.

Frobenius‡ has determined by another method the group-characters of the

*The abstract of the above paper appeared in the *Bulletin of the American Mathematical Society*, April, 1904. Just recently Schur has computed by different methods the characters of these same types of groups: Untersuchungen über die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutionen, Crelle, Vol. 132, 1906–7, (Heft 2).

†Berliner Sitzungsberichte, 1896, pp. 985-1021.

‡Ueber die Composition der Charaktere einer Gruppe, Berliner Sitzungsberichte, 1899, pp. 330-339.

groups $SLH(2, 3)$, $(2, 5)$, and of the alternating group on six letters of order 360, which is isomorphic with the group $LF(2, 3^2)$ of determinant unity. Burnside* has obtained the group-characters of the binary linear homogeneous group in the $GF[2^3]$ of order 504. The results in this paper agree with those for the above special groups.

I.

The Binary Linear Homogeneous Group H in the $GF[p^n]$, $p > 2$, of Determinant Unity.

§ 1.

The order of the group H is $h = p^n(p^n - 1)$.† For the substitution

$$R: \quad \begin{array}{l} x' = \alpha x + \beta y, \\ y' = \gamma x + \delta y, \end{array} \qquad \alpha\delta - \beta\gamma = 1,$$

we use the notation‡ $R = \begin{pmatrix} \alpha, & \beta \\ \gamma, & \delta \end{pmatrix}$.

We first reduce the substitutions of H to their canonical forms.§ For this purpose we consider the characteristic equation

$$K^2 - K(\alpha + \delta) + 1 = 0 \tag{1}$$

of the substitution $\begin{pmatrix} \alpha, & \beta \\ \gamma, & \delta \end{pmatrix}$. If the roots of this equation are distinct we get the canonical form A or B:

$$A: \quad \begin{pmatrix} \rho, & 0 \\ 0, & \rho^{-1} \end{pmatrix}, \quad \rho \text{ a mark} \neq 0 \text{ of the } GF[p^n],$$

$$B: \quad \begin{pmatrix} \sigma, & 0 \\ 0, & \sigma^{-1} \end{pmatrix}, \quad \sigma \text{ a root of } \sigma^{s+1} = 1,$$

according as the equation (1) is reducible or irreducible in the $GF[p^n]$. If the

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* Proc. Lond. Math. Soc., Series 2, Vol. I—Part 2, p. 116.

† We shall throughout denote p^n by s, except in the notation $GF[p^n]$.

‡ For the substitution R taken fractionally we use the notation $R = \left(\frac{\alpha, \ \beta}{\gamma, \ \delta}\right)$. To the two substitutions $\begin{pmatrix} \alpha, & \beta \\ \gamma, & \delta \end{pmatrix}$ and $\begin{pmatrix} -\alpha, & -\beta \\ -\gamma, & -\delta \end{pmatrix}$ of H corresponds the one substitution $\left(\frac{\alpha, \ \beta}{\gamma, \ \delta}\right)$ of F. We have therefore a two-to-one correspondence between H and F.

§ Dickson Linear Groups, §§ 214-216, 225.

roots of (1) are equal they must be $+1$ or -1. We obtain then (possibly by a transformation of determinant not unity) the canonical form

$$C:\quad \begin{pmatrix} \pm 1, & \pm 1 \\ 0, & \pm 1 \end{pmatrix}.$$

We define $\kappa = \frac{1}{2}(\alpha + \delta)$ as the invariant* of the substitution $\begin{pmatrix} \alpha, & \beta \\ \gamma, & \delta \end{pmatrix}$. If two substitutions of H have the same invariant, they have the same characteristic equation, and therefore the same canonical form. If two substitutions U and V of H have the same canonical form, there exists† a binary linear homogeneous substitution W belonging to the $GF[p^n]$ (but not necessarily of determinant unity) such that $U = W^{-1} V W$. Precisely as in §225 (Dickson, Linear Groups) we can prove that if U and V have the same canonical form A or B, there exists a substitution W_1 of H which transforms U into V; also that every substitution of H of invariant ± 1 (except $\begin{pmatrix} -1, & 0 \\ 0, & -1 \end{pmatrix}$ and the identity) is conjugate within H to one or other of the types

$$C_0:\quad \begin{pmatrix} \pm 1, & \pm 1 \\ 0, & \pm 1 \end{pmatrix},$$

$$C_1:\quad \begin{pmatrix} \pm 1, & \pm \mu \\ 0, & \pm 1 \end{pmatrix},$$

where μ is a particular not-square in the $GF[p^n]$; and further that the two types C_0, C_1 are not conjugate within H. Hence we have the result:

Two substitutions of H having the same invariant (not ± 1) are conjugate within H.

A) Let ρ be a primitive root of the $GF[p^n]$. The substitution $R = \begin{pmatrix} \rho, & 0 \\ 0, & \rho^{-1} \end{pmatrix}$ is of period $s-1$. To study the conjugacy of the substitutions $R^a = \begin{pmatrix} \rho^a, & 0 \\ 0, & \rho^{-a} \end{pmatrix}$ we transform R^a by $U = \begin{pmatrix} \alpha, & \beta \\ \gamma, & \delta \end{pmatrix}$, $\alpha\delta - \beta\gamma = 1$, and obtain

$$V = U^{-1} R^a U = \begin{pmatrix} \alpha\delta\rho^a - \beta\gamma\rho^{-a}, & -\alpha\beta(\rho^a - \rho^{-a}) \\ \gamma\delta(\rho^a - \rho^{-a}), & -\beta\gamma\rho^a + \alpha\delta\rho^{-a} \end{pmatrix}.$$

In order that V shall be identical with R^a (*i. e.*, U commutative with R^a) it is

* In the case of F we define $\kappa = \pm \frac{1}{2}(\alpha + \delta)$ as the invariant of the substitution $\begin{pmatrix} \alpha, & \beta \\ \gamma, & \delta \end{pmatrix}$.

† Dickson, Linear Groups, § 216.

necessary that either $\alpha\beta=\gamma\delta=0$, or $\rho^{a}-\rho^{-a}=0$. The first alternative leads to two cases:

$$1)\quad \text{if } \beta=\gamma=0 \text{ then } V=\begin{pmatrix}\rho^{a}, & 0\\ 0, & \rho^{-a}\end{pmatrix}=R^{a},\quad U=\begin{pmatrix}\alpha, & 0\\ 0, & \alpha^{-1}\end{pmatrix};$$

$$2)\quad \text{if } \alpha=\delta=0 \text{ then } V=\begin{pmatrix}\rho^{-a}, & 0\\ 0, & \rho^{-a}\end{pmatrix}=R^{-a},\quad U=\begin{pmatrix}0, & \beta\\ -\beta^{-1}, & 0\end{pmatrix}.$$

If $R^{a}\ddagger R^{-a}$ we have $s-1$ substitutions U commutative with R^{a}; and therefore R^{a} is one of $s(s^{2}-1)\div(s-1)=s(s+1)$ conjugate substitutions. If $R^{a}=R^{-a}$ then $\rho^{a}=\rho^{-a}$, which is the second alternative. According as $\rho^{a}=+1$ or -1, R^{a} is the identity or $R^{\frac{s-1}{2}}\equiv\begin{pmatrix}-1, & 0\\ 0, & -1\end{pmatrix}$; each of these substitutions is conjugate only to itself. With the exception of these the powers of R are conjugate in pairs, thus representing $\frac{1}{2}(s-3)$ classes of conjugate substitutions, each class containing $s(s+1)$ substitutions.*

B). The group H is holoedrically isomorphic with the group† $G_{2,p,n}$ of substitutions

$$U=\begin{pmatrix}A, & B\\ -\bar{B}, & \bar{A}\end{pmatrix},\quad (A\bar{A}+B\bar{B}=1),$$

where $\bar{A}\equiv A^{s}$ is the conjugate of A with respect to the $GF[p^{n}]$. If σ is a primitive root of the equation $\sigma^{s+1}=1$, so that $\bar{\sigma}=\sigma^{-1}$, then the substitution $S=\begin{pmatrix}\sigma, & 0\\ 0, & \sigma^{-1}\end{pmatrix}$ is of period $s+1$. As above we find that the powers of S, except $S^{\frac{s+1}{2}}=\begin{pmatrix}-1, & 0\\ 0, & -1\end{pmatrix}$ and S^{s+1}, which is the identity, are conjugate in pairs, viz., S^{b} with S^{-b}. There are $\frac{1}{2}(s-1)$ classes represented by the powers of S, each containing $s(s-1)$ substitutions.‡

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*In the case of the group F if $R^{a}=R^{-a}$ either $\rho^{a}=\rho^{-a}$, *i. e.*, R^{a} is the identity, or $\rho^{a}=-\rho^{-a}$, which is possible only if $s-1$ is divisible by 4; $R^{a}\equiv R^{\frac{s-1}{4}}$ is commutative with $s-1$ substitutions U, and is therefore one of $\frac{1}{2}s(s+1)$ conjugate substitutions. If we define ε as $+1$ or -1 according as s has the form $4l+1$ or $4l-1$, where l is an integer, then we have $\frac{1}{2}(s-2+\varepsilon)$ classes represented by the powers of R, each containing $(ss+1)$ substitutions except the class of period two, which contains $\frac{1}{2}s(s+1)$ substitutions.

†Dickson, Linear Groups, p. 132.

‡The substitution $S=\left(\frac{\sigma,\ 0}{0,\ \sigma^{-1}}\right)$ of F is of period $\frac{1}{2}(s+1)$; the substitutions S^{b} (not the identity) are conjugate in pairs except when $\varepsilon=-1$, and then $S^{\frac{s+1}{4}}$ is conjugate only to itself and is of period two. We have $\frac{1}{2}(s-\varepsilon)$ classes, each containing $s(s-1)$ substitutions, except the class of period two, which contains $\frac{1}{2}s(s-1)$ substitutions.

The numbers $\pm a$ $(\pm b)$ taken mod. $s-1$ (mod. $s+1$) will be called indifferently the index of the class represented by R^a (S^b). We have defined $x=\frac{1}{2}(\alpha+\delta)$ as the invariant of the substitution $\begin{pmatrix}\alpha, & \beta \\ \gamma, & \delta\end{pmatrix}$. The substitutions* R^a (S^b) are characterized by the property that x^2-1 is a square (not-square) in the $GF[p^n]$.

C) Consider the substitution

$$T_\mu = \begin{pmatrix}1, & \mu \\ 0, & 1\end{pmatrix}, \ \mu \text{ a mark} \neq 0 \text{ of the } GF[p^n].$$

Transforming T_μ by $U=\begin{pmatrix}\alpha, & \beta \\ \gamma, & \delta\end{pmatrix}$, $\alpha\delta-\beta\gamma=1$, we obtain

$$V=U^{-1}\,T_\mu\,U=\begin{pmatrix}1-\alpha\gamma\mu, & \alpha^2\mu \\ -\gamma^2\mu, & 1+\alpha\gamma\mu\end{pmatrix}.$$

T_μ is commutative only with those substitutions U for which $\gamma=0$, $\alpha=\pm 1$, in number $2s$; hence T_μ is one of $s(s^2-1)\div 2s=\frac{1}{2}(s^2-1)$ conjugate substitutions. We observe that the conjugate substitutions T_μ and V have the property that μ and $\alpha^2\mu$ $(\alpha\neq 0)$, or μ and $\gamma^2\mu$ in case $\alpha=0$, are both squares or both not-squares in the $GF[p^n]$. This condition can easily be proved to be sufficient for the conjugacy of T_μ and V; *i.e.*, a substitution $Q=\begin{pmatrix}\alpha', & \beta' \\ \gamma', & \delta'\end{pmatrix}$, $\alpha'\delta'-\beta'\gamma'=1$, of invariant unity is conjugate to T_μ if μ and β' $(\neq 0)$, or μ and $-\gamma'$ in case $\beta'=0$, are both squares or both not-squares in the $GF[p^n]$.

The substitutions $\begin{pmatrix}\alpha, & \beta \\ \gamma, & \delta\end{pmatrix}$ (except the identity) of invariant $+1$ will be said to belong to the class (μ) or to the class (ν) according as β $(\neq 0)$, or $-\gamma$ if $\beta=0$, is a square or a not-square in the $GF[p^n]$. Similarly the substitutions $\begin{pmatrix}\alpha, & \beta \\ \gamma, & \delta\end{pmatrix}$, except $\begin{pmatrix}-1, & 0 \\ 0, & -1\end{pmatrix}$, of invariant -1 will be said to belong to the class† (m) or to the class (n) according as β $(\neq 0)$, or $-\gamma$ if $\beta=0$, is a square or a not-square in the $GF[p^n]$.

* The substitutions of period two have the invariant zero.

† To the two classes (μ) and (m) of H corresponds the one class (μ) of F; and similarly for (ν) and (n).

The substitutions* $\begin{pmatrix}1, & 0\\ 0, & 1\end{pmatrix}$ and $\begin{pmatrix}-1, & 0\\ 0, & -1\end{pmatrix}$ will be denoted by (λ) and (l) respectively.

The total number † of classes of conjugate substitutions is $\frac{1}{2}(s-3)+\frac{1}{2}(s-1)+4+2=s+4$.

Since the powers of R and S were shown to be conjugate with their reciprocals, a substitution W and its reciprocal W^{-1} belong to the same class except when W belongs to one of the classes (μ), (ν), (m), (n).

§ 2.

We define ε_κ to be $+1$ or -1 according as κ^2-1 is a square or a not-square in the $GF[p^n]$, where κ denotes henceforth any invariant except ± 1. In place of ε_0 we write ε.‡

The class whose invariant is κ is denoted by (κ). This notation is unique. For, two conjugate substitutions have the same invariant; and it has been proved that if two substitutions of H have the same invariant (not ± 1) they are conjugate. Instead of κ we shall nearly always use $\alpha, \beta, \gamma, \ldots.$, and we shall denote the indices of the classes $(\alpha), (\beta), (\gamma), \ldots.$ by $\pm a, \pm b, \pm c, \ldots.$ respectively. These indices are taken mod. $s-1$ or mod. $s+1$ according as κ^2-1 is a square or a not-square in the $GF[p^n]$.

Denoting by h_θ the number of substitutions in a class (θ) we have §

$$h_\lambda=h_l=1,\ h_\mu=h_\nu=h_m=h_n=\tfrac{1}{2}(s^2-1),\ h_\kappa=s(s+\varepsilon_\kappa).$$

If $\varepsilon_\alpha=\varepsilon$, $\varepsilon_\beta=-\varepsilon$, the numbers‖ of classes (α) and (β) are $\frac{1}{2}(s-2-\varepsilon)$ and $\frac{1}{2}(s-2+\varepsilon)$ respectively.

We define¶ ζ_κ as $+1$ or -1 according as $-2(1-\kappa)$ is a square or a not-square in the $GF[p^n]$. Then $\zeta_\alpha=\varepsilon_\alpha(-1)^a$.

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* The one corresponding substitution $\begin{pmatrix}1, & 0\\ 0, & -1\end{pmatrix}$ of F will be denoted by (λ).

† For F the total number of classes is $\frac{1}{2}(s+5)$.

‡ For the definition of ε compare p. 390, foot-note*.

§ For F we have $h_\lambda=1$, $h_\mu=h_\nu=\frac{1}{2}(s^2-1)$, $h_\kappa=s(s+\varepsilon_\kappa)$, $h_0=\frac{1}{2}s(s+\varepsilon)$. The class (0) requires to be distinguished from the other classes (κ) more frequently in the case of F than in the case of H.

‖ For F the numbers of classes (α) and (β) are $\frac{1}{2}(s-\varepsilon)$ and $\frac{1}{2}(s-2+\varepsilon)$ respectively.

¶ In the case of F we define η_κ as $+1$, -1, or 0 according as $-2(1+\kappa)$ and $-2(1-\kappa)$ are both squares, both not-squares, or one a square and the other a not-square, in the $GF[p^n]$. We also define $2\eta_1=\varepsilon$. If $\varepsilon_\alpha=\varepsilon$ then $\eta_\alpha=(-1)^a\varepsilon$; if $\varepsilon_\beta=-\varepsilon$ then $\eta_\beta=0$; further, $\eta_\kappa=\frac{1}{2}(1+\varepsilon\varepsilon_\kappa)\zeta_\kappa$.

§ 3.

Three (distinct or equal) classes (α), (β), (γ) are called *concordant* if between their invariants there exists the relation*

$$\alpha^2 + \beta^2 + \gamma^2 - 2\alpha\beta\gamma = 1; \tag{1}$$

otherwise they are called *discordant*. If we write (1) in the form

$$(\alpha^2 - 1)(\beta^2 - 1) = (\alpha\beta - \gamma)^2$$

it follows that $\varepsilon_\alpha = \varepsilon_\beta$; similarly $\varepsilon_\beta = \varepsilon_\gamma$. Hence three concordant classes must all be represented by powers of R or all by powers of S. If α and β are given we find that the classes whose invariants are

$$\gamma = \alpha\beta + \sqrt{(\alpha^2 - 1)(\beta^2 - 1)}, \qquad \delta = \alpha\beta - \sqrt{(\alpha^2 - 1)(\beta^2 - 1)} \tag{2}$$

are concordant with (α) and (β). If $\beta = \alpha$ then $\gamma = 2\alpha^2 - 1$; and therefore we have $\varepsilon_\alpha = \varepsilon_{2\alpha^2} - 1$.

Let r denote ρ or σ according as $\varepsilon_\alpha = +1$ or -1. Substituting the values $2\alpha = r^a + r^{-a}$, etc., in (1), and factoring, we obtain

$$(r^{a+b+c} - 1)(r^{-a-b+c} - 1)(r^{-a+b-c} - 1)(r^{a-b-c} - 1) = 0.$$

Hence $a \pm b \pm c \equiv 0$ (mod. $s - 1$ or $s + 1$ according as $\varepsilon_\alpha = +1$ or -1). The indices of the two classes (γ) and (δ) concordant with (α) and (β) are therefore

$$c \equiv a + b, \quad d \equiv a - b \,(\text{mod. } s - 1,\ s + 1 \text{ respectively}).$$

§ 4.

Let Θ, Φ, Ψ represent the substitutions of any three distinct or equal classes (θ), (ϕ), (ψ) respectively, and let (θ'), (ϕ'), (ψ') denote the classes of the inverse substitutions Θ^{-1}, Φ^{-1}, Ψ^{-1} respectively. If Θ, Φ, Ψ run through all the substitutions of their respective classes, we denote† by $h_{\theta\phi\psi}$ the number of times we obtain the relation $\Theta\Phi\Psi = E$ (the identity), or $\Theta\Phi = \Psi^{-1}$. The subscripts θ, ϕ, ψ may be permuted in any manner.

To obtain $h_{\alpha\beta\gamma}$ we determine $\dfrac{h_{\alpha\beta\gamma}}{h_\alpha}$; we take a *particular* substitution of (α), compound it with all the substitutions of (β),

$$\begin{pmatrix}\alpha, & 1\\ \alpha^2 - 1, & \alpha\end{pmatrix}\begin{pmatrix}\xi, & \eta\\ \zeta, & 2\beta - \xi\end{pmatrix} = \begin{pmatrix}\alpha\xi + \eta(\alpha^2 - 1), & \xi + \alpha\eta\\ \alpha\zeta + (2\beta - \xi)(\alpha^2 - 1), & \zeta + \alpha(2\beta - \xi)\end{pmatrix},$$

and determine how many of the resulting substitutions belong to the class (γ').

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* For F this relation takes the form $\alpha^2 + \beta^2 + \gamma^2 \pm 2\alpha\beta\gamma = 1$.

† Frobenius, Über Gruppencharaktere, 1896, pp. 987, 988.

In order that the resulting substitutions may be of class $(\gamma') \equiv (\gamma)$* and have the determinant unity, we must have

$$\alpha\xi + \eta(\alpha^2 - 1) + \zeta + \alpha(2\beta - \xi) = 2\gamma,$$
$$\xi(2\gamma - \xi) - \eta\zeta = 1.$$

The number of distinct sets of solutions ξ, η, ζ of these equations will give $\frac{h_{\alpha\beta\gamma}}{h_\alpha}$. Eliminating ζ we obtain

$$(\xi - \beta)^2 - (\alpha^2 - 1)\left(\eta + \frac{\alpha\beta - \gamma}{\alpha^2 - 1}\right)^2 = \frac{(\alpha^2 - 1)(\beta^2 - 1) - (\alpha\beta - \gamma)^2}{\alpha^2 - 1}.$$

If (α), (β), (γ) are discordant the right-hand side is distinct from zero, and we obtain $s - \varepsilon_\alpha$ sets of solutions†. If (α), (β), (γ) are concordant‡ the right-hand side is zero, and we obtain $s + \varepsilon_\alpha(s - 1)$ sets of solutions. Hence $h_{\alpha\beta\gamma} = h + \varepsilon_\alpha s^2(s + \varepsilon_\alpha)$ or h according as (α), (β), (γ) are concordant or discordant.

If we denote the substitutions of (μ) and (ν) by P and Q respectively, then according as $\varepsilon = 1$ or -1 will P^{-1} belong to (μ) or (ν), and Q^{-1} to (ν) or (μ) Hence $h_{\lambda\mu\nu} = \frac{1}{2} h_\mu (1 - \varepsilon)$. Similarly $h_{\lambda mn} = \frac{1}{2} h_\mu (1 - \varepsilon)$.

The group H is self-conjugate under the group of all binary linear homogeneous substitutions of determinant $\neq 0$; by a substitution of determinant a not-square in the $GF[p^n]$ the class (μ) is transformed into (ν), and (ν) into (μ), and simultaneously (m) into (n) and (n) into (m). Hence the notations (μ) and (ν) are interchangeable; likewise (m) and (n); furthermore the interchange of (μ) and (ν) must be accompanied by the interchange of (m) and (n), and vice versa.

To determine $h_{\mu\mu\nu}$ we compute $h_\mu(h_{\mu\nu\mu} + h_{\mu\nu\nu} + h_{\mu\nu\lambda})$; we take a definite substitution of (μ), compound it with all the substitutions of (ν),

$$\begin{pmatrix} 0, & 1 \\ -1, & 2 \end{pmatrix}\begin{pmatrix} \xi, & \eta \\ \zeta, & 2-\xi \end{pmatrix} = \begin{pmatrix} -\eta, & \xi + 2\eta \\ -2+\xi, & \zeta + 4 - 2\xi \end{pmatrix},$$

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* See last paragraph of §1.

† Dickson, Linear Groups, p. 46.

‡ In the case of F we have the equations

$$(\xi - \beta)^2 - (\alpha^2 - 1)\left(\eta + \frac{\alpha\beta \pm \gamma}{\alpha^2 - 1}\right)^2 = \frac{(\alpha^2 - 1)(\beta^2 - 1) - (\alpha\beta \pm \gamma)^2}{\alpha^2 - 1}.$$

If (α), (β), (γ) are concordant then one of the relations $(\alpha^2 - 1)(\beta^2 - 1) - (\alpha\beta \pm \gamma)^2 = 0$ holds and not the other. Suppose that $(\alpha^2 - 1)(\beta^2 - 1) - (\alpha\beta + \gamma)^2 = 0$; then the equations with the upper and lower signs have $s + \varepsilon_\alpha(s - 1)$ and $s - \varepsilon_\alpha$ sets of solutions respectively; in all $2(s - \varepsilon_\alpha) + \varepsilon_\alpha s$ sets of solutions.

and find how many of the resulting substitutions belong to the classes (λ), (μ), (ν) collectively, *i. e.*, have the invariant $+1$. We have therefore the equations

$$\xi(2-\xi)-\eta\zeta=1,$$
$$-\eta+\zeta+4-2\xi=2.$$

Eliminating ξ we get $(\eta+\zeta)^2=0$, or $-\zeta=\eta$. We can take for η every not-square in the $GF[p^n]$, thus obtaining $\frac{1}{2}(s-1)$ sets of solutions. But $h_{\mu\nu\mu}=h_{\nu\mu\nu}=h_{\mu\nu\nu}$; hence $h_{\mu\mu\nu}=\frac{1}{4}h_\mu(s-2+\varepsilon)$.

Proceeding in this way we get the following results.*

$h_{\alpha\beta\gamma}=h+\varepsilon_\alpha s^2(s+\varepsilon_\alpha)$ or h according as (α), (β), (γ) are concordant or discordant,

$h_{\alpha\alpha\lambda}=h_\alpha,\ h_{\alpha\alpha l}=0,\ h_{\alpha\alpha\mu}=h_{\alpha\alpha\nu}=\frac{1}{2}h(1+\varepsilon_\alpha),\ h_{\alpha\alpha m}=h_{\alpha\alpha n}=\frac{1}{2}h$ if $\alpha\neq 0$,

$h_{oom}=h_{oon}=\frac{1}{2}h(1+\varepsilon),\ h_{-\alpha\alpha l}=h_\alpha,\ h_{-\alpha\alpha m}=\frac{1}{2}h(1+\varepsilon_\alpha)$,

$h_{\alpha\beta\lambda}=0,\ h_{\alpha\beta\mu}=h_{\alpha\beta\nu}=h_{\alpha\beta m}=h_{\alpha\beta n}=\frac{1}{2}h$,

$h_{\alpha\lambda l}=h_{\alpha\lambda\mu}=h_{\alpha\lambda\nu}=h_{\alpha\lambda m}=h_{\alpha\lambda n}=h_{\alpha l\mu}=h_{\alpha l\nu}=h_{\alpha lm}=h_{\alpha ln}=0$,

$h_{\alpha\mu\mu}=h_{\alpha\nu\nu}=h_{\alpha mm}=h_{\alpha nn}=\frac{1}{4}h(1+\varepsilon\zeta_\alpha),\ h_{\alpha\mu\nu}=h_{\alpha mn}=\frac{1}{4}h(1-\varepsilon\zeta_\alpha)$,

$h_{\alpha\mu m}=h_{\alpha\nu n}=\frac{1}{4}h(1+\varepsilon_\alpha\zeta_\alpha),\ h_{\alpha m\nu}=h_{\alpha\mu n}=\frac{1}{4}h(1-\varepsilon_\alpha\zeta_\alpha)$,

$h_{l\mu m}=h_{l\nu n}=h_{\lambda\mu\mu}=h_{\lambda\nu\nu}=h_{\lambda mm}=h_{\lambda nn}=\frac{1}{2}h_\mu(1+\varepsilon)$,

$h_{\lambda m\nu}=h_{\lambda\mu n}=h_{\lambda\mu m}=h_{\lambda\nu n}=h_{l\mu\mu}=h_{l\nu\nu}=h_{l\mu\nu}=h_{lmn}=0$,

$h_{lmm}=h_{lnn}=h_{\mu\mu n}=h_{m\nu\nu}=h_{\mu m\nu}=h_{\mu\nu n}=0$,

$h_{mmn}=h_{mnn}=0,\ h_{\lambda\mu\nu}=h_{\lambda mn}=h_{l\mu n}=h_{lm\nu}=\frac{1}{2}h_\mu(1-\varepsilon)$,

$h_{\mu\mu\mu}=h_{\nu\nu\nu}=h_{\mu mm}=h_{\nu nn}=\frac{1}{4}h_\mu(s-2-3\varepsilon)$,

$h_{\mu\mu\nu}=h_{\mu\nu\nu}=h_{\mu nn}=h_{mm\nu}=h_{\mu mn}=h_{m\nu n}=\frac{1}{4}h_\mu(s-2+\varepsilon)$,

$h_{\mu\mu m}=h_{\nu\nu n}=h_{mmm}=h_{nnn}=s\,h_\mu$.

§ 5.

The value of a group-character χ for any class (θ) is denoted by χ_θ; but sometimes the value of a group-character for a class represented by a power of R or S is denoted by $\chi(R^a)$ or $\chi(S^b)$ respectively. Also $\chi_\lambda=f$.

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* These results reduce to a very few in the case of F, since $(l)=(\lambda)$, $(m)=(\mu)$, $(n)=(\nu)$. For F we have the following additional results: $h_{o\alpha\beta}=2h+\varepsilon s^2(s+\varepsilon)$ or $2h$ according as (o), (α), (β) are concordant or discordant; $h_{ooo}=\frac{1}{2}h$, $h_{oo\alpha}=h$, $h_{oo\lambda}=\frac{1}{2}s(s+\varepsilon)$, $h_{oo\mu}=h_{oo\nu}=\frac{1}{2}h(1+\varepsilon)$, $h_{o\alpha\mu}=h_{o\alpha\nu}=h$, $h_{o\mu\mu}=h_{o\nu\nu}=\frac{1}{2}h(1+\varepsilon\eta)$, $h_{o\mu\nu}=\frac{1}{2}h(1-\varepsilon\eta)$, where $\eta=\eta_o$.

We make the following abbreviations:*

$$x = \tfrac{1}{2}(\chi_\mu + \chi_\nu + \chi_m + \chi_n) + \sum_\kappa \chi_\kappa,$$

$$y = \tfrac{\epsilon}{4}(\chi_\mu + \chi_\nu + \chi_m + \chi_n) + \sum_\kappa \zeta_\kappa \chi_\kappa,$$

$$z = \tfrac{\epsilon}{4}(\chi_\mu + \chi_\nu + \chi_m + \chi_n) + \sum_\kappa \varepsilon_\kappa \zeta_\kappa \chi_\kappa.$$

From the relation †

$$h_\theta h_\psi \chi_\theta \chi_\psi = f \sum_\phi h_{\theta' \phi . \psi} \chi_\phi,$$

where (θ), (ϕ), (ψ) are any three classes, we derive the following set of equations ‡

$$\chi_1^2 = f^2, \tag{1}$$

$$\chi_\alpha \chi_1 = f \chi_{-\alpha}, \tag{2}$$

$$s \chi_\alpha \chi_\beta = f x, \quad (\varepsilon_\beta = -\varepsilon_\alpha), \tag{3}$$

$$\frac{s(s+\varepsilon_\alpha)}{f} \chi_\alpha \chi_\beta = x(s-\varepsilon_\alpha) + \varepsilon_\alpha s (\chi_\gamma + \chi_\delta), \quad (\varepsilon_\beta = \varepsilon_\alpha, \ \alpha \neq -\beta), \tag{4}$$

where γ and δ are determined by (3), §3,

$$\frac{s(s+\varepsilon_\alpha)}{f} \chi_\alpha \chi_{-\alpha} = \chi_1 + x(s-\varepsilon_\alpha) + \tfrac{1}{2}\varepsilon_\alpha (s-\varepsilon_\alpha)(\chi_m + \chi_n) + \varepsilon_\alpha s \chi_{-(2\alpha^2-1)}, \tag{5}$$

$$\frac{s(s+\varepsilon_\alpha)}{f} \chi_\alpha^2 = f + x(s-\varepsilon_\alpha) + \tfrac{1}{2}\varepsilon_\alpha (s-\varepsilon_\alpha)(\chi_\mu + \chi_\nu) + \varepsilon_\alpha s \chi_{2\alpha^2-1}, \tag{6}$$

$$\frac{s(s+\varepsilon)}{f} \chi_0^2 = f + x(s-\varepsilon) + \tfrac{1}{2}\varepsilon (s-\varepsilon)(\chi_\mu + \chi_\nu + \chi_m + \chi_n), \tag{7}$$

$$\frac{s+\varepsilon_\alpha}{f} \chi_\alpha \chi_\mu = x + \varepsilon_\alpha \chi_\alpha + \tfrac{1}{2}\zeta_\alpha (\chi_\mu - \chi_\nu) + \tfrac{1}{2}\varepsilon\varepsilon_\alpha \zeta_\alpha (\chi_m - \chi_n), \tag{8}$$

$$\frac{s+\varepsilon_\alpha}{f} \chi_\alpha \chi_\nu = x + \varepsilon_\alpha \chi_\alpha - \tfrac{1}{2}\zeta_\alpha (\chi_\mu - \chi_\nu) - \tfrac{1}{2}\varepsilon\varepsilon_\alpha \zeta_\alpha (\chi_m - \chi_n), \tag{8a}$$

$$\frac{s+\varepsilon_\alpha}{f} \chi_\alpha \chi_m = x + \varepsilon_\alpha \chi_{-\alpha} + \tfrac{1}{2}\varepsilon\varepsilon_\alpha \zeta_\alpha (\chi_\mu - \chi_\nu) + \tfrac{1}{2}\zeta_\alpha (\chi_m - \chi_n), \tag{9}$$

$$\frac{s+\varepsilon_\alpha}{f} \chi_\alpha \chi_n = x + \varepsilon_\alpha \chi_{-\alpha} - \tfrac{1}{2}\varepsilon\varepsilon_\alpha \zeta_\alpha (\chi_\mu - \chi_\nu) - \tfrac{1}{2}\zeta_\alpha (\chi_m - \chi_n), \tag{9a}$$

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*In the case of F we make the abbreviations

$$\left.\begin{aligned} x &= \chi_0 + \chi_\mu + \chi_\nu + 2\sum_\kappa \chi_\kappa, \\ y &= \eta_0 \chi_0 + \eta_1 (\chi_\mu + \chi_\nu) + 2\sum_\kappa \eta_\kappa \chi_\kappa. \end{aligned}\right\} \quad \kappa \neq 0,$$

† Über Gruppencharaktere, p. 994.

‡ We get the equations for F from (3), (4), (6)—(8*a*), (10)—(10*b*) if we set $\chi_m = \chi_\mu$, $\chi_n = \chi_\nu$, and remember that $\frac{1}{2}(1 + \varepsilon\varepsilon_\alpha)\zeta_\alpha = \eta_\alpha$.

$$\frac{s^2-1}{fs}\chi_\mu^2 = x+\varepsilon y+(1+\varepsilon)\frac{f}{s}-\frac{s+2\varepsilon+4}{4s}(\chi_\mu+\chi_\nu) -\frac{1}{s}(\chi_\mu-\chi_\nu)+\tfrac{1}{4}(\chi_m+\chi_n)+\varepsilon(\chi_m-\chi_n), \tag{10}$$

$$\frac{s^2-1}{fs}\chi_\nu^2 = x+\varepsilon y+(1+\varepsilon)\frac{f}{s}-\frac{s+2\varepsilon+4}{4s}(\chi_\mu+\chi_\nu)+\frac{1}{s}(\chi_\mu-\chi_\nu) +\tfrac{1}{4}(\chi_m+\chi_n)-\varepsilon(\chi_m-\chi_n), \tag{10a}$$

$$\frac{s^2-1}{fs}\chi_\mu\chi_\nu = x-\varepsilon y+(1-\varepsilon)\frac{f}{s}+\frac{s+2\varepsilon-4}{4s}(\chi_\mu+\chi_\nu)-\tfrac{1}{4}(\chi_m+\chi_n), \tag{10b}$$

$$\frac{s^2-1}{fs}\chi_\mu\chi_m = x+z+\frac{1+\varepsilon}{s}\chi_l+\frac{2-\varepsilon}{4}(\chi_\mu+\chi_\nu)+\varepsilon(\chi_\mu-\chi_\nu) -\frac{\varepsilon s+2\varepsilon+4}{4s}(\chi_m+\chi_n)-\frac{1}{s}(\chi_m-\chi_n), \tag{11}$$

$$\frac{s^2-1}{fs}\chi_\nu\chi_n = x+z+\frac{1+\varepsilon}{s}\chi_l+\frac{2-\varepsilon}{4}(\chi_\mu+\chi_\nu)-\varepsilon(\chi_\mu-\chi_\nu) -\frac{\varepsilon s+2\varepsilon+4}{4s}(\chi_m+\chi_n)+\frac{1}{s}(\chi_m-\chi_n), \tag{11a}$$

$$\frac{s^2-1}{fs}\chi_\mu\chi_n = \frac{s^2-1}{fs}\chi_m\chi_\nu = x-z+\frac{1-\varepsilon}{s}\chi_l -\frac{2-\varepsilon}{4}(\chi_\mu+\chi_\nu)+\frac{\varepsilon s+2\varepsilon-4}{4s}(\chi_m+\chi_n), \tag{11b}$$

$$\chi_l\chi_\mu = f\chi_m, \tag{12}$$

$$\chi_l\chi_\nu = f\chi_n. \tag{12a}$$

I. We seek first those solutions for which χ_μ and χ_ν are distinct. Then, according to (12) and (12a), χ_m and χ_n are distinct. From (1) we find $\chi_l=\pm f$.

Suppose first that $\chi_l=f$. Then $\chi_m=\chi_\mu$ and $\chi_n=\chi_\nu$, and from (8) and (8a) we obtain $\frac{s+\varepsilon\alpha}{f}\chi_\alpha=\zeta_\alpha(1+\varepsilon\varepsilon_\alpha)$. If $\varepsilon_\alpha=-\varepsilon$ then $\chi_\alpha=0$; if $\varepsilon_\alpha=\varepsilon$, and if we set the proportionality factor* $f=\frac{1}{2}(s+\varepsilon)$, we obtain $\chi_\alpha=\zeta_\alpha$. Hence $\chi(R^a)=\frac{(1+\varepsilon)(-1)^a}{2}$, $\chi(S^b)=-\frac{(1-\varepsilon)(-1)^b}{2}$. According to (3), $x=0$. If in (6) $\varepsilon_\alpha=-\varepsilon$, then $\chi_\mu+\chi_\nu=\varepsilon$, and $y=\frac{1}{2}+\sum_\kappa\zeta_\kappa^2$, where $\varepsilon_\kappa=\varepsilon$. The number of classes (κ) for which $\varepsilon_\kappa=\varepsilon$ is $\frac{1}{2}(s-2-\varepsilon)$, and therefore $y=\frac{1}{2}(s-1-\varepsilon)$. Similarly $z=\frac{1}{2}(s-2)$. From (10b) we get $4\chi_\mu\chi_\nu=1-\varepsilon s$;

* The proportionality factor may be chosen arbitrarily. See Über Gruppencharaktere, p. 999.

also we already have $\chi_\mu + \chi_\nu = \varepsilon$. Hence $\chi_\mu = \chi_m = \frac{1}{2}(\varepsilon \pm \sqrt{\varepsilon s})$, $\chi_\nu = \chi_n = \frac{1}{2}(\varepsilon \mp \sqrt{\varepsilon s})$. These values of the characters will be found to satisfy all the equations.

Let next $\chi_l = -f$. Then $\chi_m = -\chi_\mu$, $\chi_n = -\chi_\nu$. From (8) and (8a) we get as before $\frac{s+\varepsilon_\alpha}{f}\chi_\alpha = \zeta_\alpha(1-\varepsilon\varepsilon_\alpha)$. If $\varepsilon_\alpha = \varepsilon$, $\chi_\alpha = 0$; if $\varepsilon_\alpha = -\varepsilon$, and if we set $f = \frac{1}{2}(s-\varepsilon)$, we obtain* $\chi_\alpha = \zeta_\alpha$. According to (3), $x = 0$; if in (6) $\varepsilon_\alpha = \varepsilon$ then $\chi_\mu + \chi_\nu = -\varepsilon$, and therefore $\chi_m + \chi_n = \varepsilon$. Also $y = \frac{1}{2}(s-2+\varepsilon)$ and $z = -{}^{\textstyle '}(s-2+\varepsilon)$. From (10b) we obtain $4\chi_\mu\chi_\nu = 1 - \varepsilon s$; this combined with $\chi_\mu + \chi_\nu = -\varepsilon$ gives $\chi_\mu = -\chi_m = \frac{1}{2}(-\varepsilon \pm \sqrt{\varepsilon s})$, $\chi_\nu = -\chi_n = \frac{1}{2}(-\varepsilon \mp \sqrt{\varepsilon s})$.

II. For all other solutions $\chi_\mu = \chi_\nu$, and therefore $\chi_m = \chi_n$. Instead of equations (10)–(11b) we shall use the following which are obtained from them by addition and subtraction:

$$\frac{s^2-1}{f}\chi_\mu^2 \quad = sx + f - 2\chi_\mu, \tag{10'}$$

$$2\varepsilon s y = (s+2\varepsilon)\chi_\mu - s\chi_m - 2\varepsilon f, \tag{10''}$$

$$\frac{s^2-1}{f}\chi_\mu\chi_m = sx + \chi_l - 2\chi_m, \tag{11'}$$

$$2\varepsilon s z = s(1-2\varepsilon)\chi_\mu + (s+2)\chi_m - 2\chi_l. \tag{11''}$$

We seek first those solutions for which x is distinct from zero. According to (3) none of the characters χ_α can be zero; and $\chi_\alpha = \chi_\gamma$ if $\varepsilon_\alpha = \varepsilon_\gamma$, *i. e.*, all characters χ_α are equal for which ε_α has the same sign. Since $\chi_\alpha = \chi_{-\alpha}$ it follows from (8) and (9) that $\chi_m = \chi_\mu$, and therefore from (12) that $\chi_l = f$.

Let $\varepsilon_\alpha = \varepsilon$, $\varepsilon_\beta = -\varepsilon$. Then* $y = \varepsilon + \chi_\alpha \sum_\alpha \zeta_\alpha + \chi_\beta \sum_\beta \zeta_\beta = \varepsilon - \varepsilon\chi_\alpha$. From (6) and (8) we obtain

$$\left.\begin{aligned} \chi_\alpha &= \tfrac{1}{s}\{(s-\varepsilon)\chi_\mu + \varepsilon f\}, \\ \chi_\beta &= \tfrac{1}{s}\{(s+\varepsilon)\chi_\mu - \varepsilon f\}. \end{aligned}\right. \tag{A}$$

In the sum $x = \frac{1}{2}(\chi_\mu + \chi_\nu + \chi_m + \chi_n) + \sum_{\alpha,\beta}(\chi_\alpha + \chi_\beta)$, $\varepsilon_\alpha = \varepsilon$, $\varepsilon_\beta = -\varepsilon$, the numbers of characters χ_α, χ_β are $\frac{1}{2}(s-2-\varepsilon)$, $\frac{1}{2}(s-2+\varepsilon)$ respectively. Hence we have

$$x = 2\chi_\mu + \tfrac{1}{2}(s-2-\varepsilon)\chi_\alpha + \tfrac{1}{2}(s-2+\varepsilon)\chi_\beta. \tag{B}$$

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* From the definition of x we get $_\kappa\ \zeta_\kappa = -\varepsilon$ or 0 according as $\varepsilon_\kappa = +\varepsilon$ or $-\varepsilon$. Hence we have in general $\sum_\kappa \zeta_\kappa = -\varepsilon$.

Eliminating χ_α and χ_β from (A) and (B) we obtain

$$sx = (s^2+1)\chi_\mu - f.$$

Substituting this value of sx in $(10')$ we obtain

$$\chi_\mu(\chi_\mu - f) = 0.$$

If first $\chi_\mu = 0$, let $f = s$; then $x = y = -1$, $\chi_\alpha = \varepsilon$, $\chi_\beta = -\varepsilon$, in general $\chi_\kappa = \varepsilon_\kappa$, and therefore $\chi(R^a) = 1$, $\chi(S^b) = -1$. Also $\chi_l = s$.

If secondly $\chi_\mu = f$, let $f = 1$; then $\chi_l = \chi_\mu = \chi_m = \chi_\alpha = \chi_\beta = 1$. Also $x = s$, $y = 0$.

III. For all other solutions $x = 0$. Then (3) becomes $\chi_\alpha \chi_\beta = 0$ $\varepsilon_\beta = -\varepsilon_\alpha$. Not all the characters χ_κ can be zero. For, if they were, by giving to ε_κ in (6) the values 1 and -1 in turn we would have $\chi_\mu + \chi_\nu = 0$, and therefore $f = 0$, which is inadmissible.

According to (3) either all $\chi_\alpha = 0$ for which $\varepsilon_\alpha = 1$, or all for which $\varepsilon_\alpha = -1$. Suppose first that $\chi_\alpha = 0$ in case $\varepsilon_\alpha = -1$; and let $\chi_l = f = s + 1$. Then, since not all the characters χ_α for $\varepsilon_\alpha = 1$ can be zero, we obtain from (8) $\chi_\mu = 1$; and from (12) $\chi_m = 1$. If $\varepsilon_\alpha = \varepsilon_\beta = 1$, and therefore $\varepsilon_\gamma = \varepsilon_\delta = 1$, we obtain from (4), (5), (6)

$$\chi_\alpha \chi_\beta = \chi_\gamma + \chi_\delta, \; \chi_\alpha \chi_{-\alpha} = \chi_{-(2\alpha^2-1)} + 2, \; \chi_\alpha^2 = \chi_{2\alpha^2-1} + 2.$$

If we set $\chi_\alpha = \xi_a$, $\xi_0 = \xi_{\frac{s-1}{2}} = 2$, these equations can be combined into one:

$$\xi_a \xi_b = \xi_{a+b} + \xi_{a-b},$$

where a and b may be distinct or equal. Let r be a new unknown; if we set $\xi_1 = r + r^{-1}$ it follows from $\xi_1\xi_1 = \xi_2 + \xi_0$ that $\xi_2 = r^2 + r^{-2}$; then from $\xi_1\xi_2 = \xi_3 + \xi_1$ it follows that $\xi_3 = r^3 + r^{-3}$; in general $\xi_a = r^a + r^{-a}$. From $\xi_{\frac{s-1}{2}} = 2$ we get $r^{\frac{s-1}{2}} = 1$. We obtain then the solutions

$$\chi_l = f = s + 1, \quad \chi_\mu = \chi_\nu = \chi_m = \chi_n = 1,$$
$$\chi_\alpha = r^a + r^{-a} \text{ if } \varepsilon_\alpha = 1, \quad \chi_\beta = 0 \text{ if } \varepsilon_\beta = -1.$$

From $(10'')$, $(11'')$ we find $y = z = -1$. The above solutions satisfy the equation $x = 0$ except when $r = 1$, and the equations $y = z = -1$ except when $r = -1$; and r can be -1 only when $\varepsilon = 1$. If $\varepsilon = 1$ the equation $r^{\frac{s-1}{2}} = 1$ has $\frac{s-5}{2}$ solutions distinct from ± 1; if $\varepsilon = -1$ it has $\frac{s-3}{2}$ solutions distinct from 1; in general it has $\frac{1}{2}(s - 4 - \varepsilon)$ admissible solutions. Since r^a and r^{-a} give the same value for $r^a + r^{-a}$ these solutions go in pairs, giving $\frac{1}{4}(s - 4 - \varepsilon)$ characters.

We next let $\chi_l = -f = -(s+1)$. Then $\chi_m = -1$; and (4), (5), (6) become

$$\chi_\alpha \chi_\beta = \chi_\gamma + \chi_\delta, \quad \chi_\alpha \chi_{-\alpha} = \chi_{-(2\alpha^2-1)} - 2, \quad \chi_\alpha^2 = \chi_{2\alpha^2-1} + 2.$$

Setting $\chi_\alpha = \xi_a$, $\xi_0 = 2$, $\xi_{\frac{s-1}{2}} = -2$ we get $\xi_a \xi_b = \xi_{a+b} + \xi_{a-b}$. Let $\xi_1 = r_1 + r_1^{-1}$; then as above we obtain $\xi_a = r_1^a + r_1^{-a}$, and also $r_1^{\frac{s-1}{2}} = -1$. We have the following solutions:

$$f = s+1, \quad \chi_l = -(s+1), \quad \chi_\mu = \chi_\nu = 1, \quad \chi_m = \chi_n = -1,$$
$$\chi_\alpha = r_1^a + r_1^{-a} \text{ if } \epsilon_\alpha = 1, \quad \chi_\beta = 0 \text{ if } \epsilon_\beta = -1.$$

Now $x = 0$, $y = z = \epsilon - 1$. The above solutions satisfy $x = 0$, and also $y = z = \epsilon - 1$ except when $r_1 = -1$, which can happen only when $\epsilon = -1$. These solutions furnish $\frac{1}{4}(s - 2 + \epsilon)$ characters.

IV. Suppose finally that $\chi_\alpha = 0$ in case $\epsilon_\alpha = 1$. Let $f = s - 1$. Assuming first that $\chi_l = f$ we get $\chi_m = \chi_\mu = -1$. From (4), (5), (6) we have

$$\chi_\alpha \chi_\beta = -\chi_\gamma - \chi_\delta, \quad \chi_\alpha \chi_{-\alpha} = -\chi_{-(2\alpha^2-1)} + 2, \quad \chi_\alpha^2 = -\chi_{2\alpha^2-1} + 2.$$

Setting $\chi_\alpha = -\xi_a$, $\xi_0 = \xi_{\frac{s+1}{2}} = 2$, we obtain $\xi_a \xi_b = \xi_{a+b} + \xi_{a-b}$. If $\xi_1 = t + t^{-1}$ then $\xi_b = t^b + t^{-b}$, and $t^{\frac{s+1}{2}} = 1$. We have then the following solutions:

$$\chi_l = f = s - 1, \quad \chi_\mu = \chi_\nu = \chi_m = \chi_n = -1,$$
$$\chi_\alpha = 0 \text{ if } \epsilon_\alpha = 1, \quad \chi_\beta = -(t^b + t^{-b}) \text{ if } \epsilon_\beta = -1.$$

The equation $x = 0$ is satisfied except when $t = 1$; and the equations $y = -1$, $z = 1 - 2\epsilon$ are satisfied except when $t = -1$, which can happen only when $\epsilon = -1$. These solutions furnish $\frac{1}{4}(s - 2 + \epsilon)$ characters.

Assuming next that $\chi_l = -f$ we get $\chi_\mu = -1$, $\chi_m = 1$; also

$$\chi_\alpha \chi_\beta = -\chi_\gamma - \chi_\delta, \quad \chi_\alpha \chi_{-\alpha} = -\chi_{-(2\alpha^2-1)} - 2, \quad \chi_\alpha^2 = -\chi_{2\alpha^2-1} + 2.$$

Setting $\chi_\alpha = -\xi_a$, $\xi_0 = 2$, $\xi_{\frac{s+1}{2}} = -2$, we obtain $\xi_a \xi_b = \xi_{a+b} + \xi_{a-b}$. If $\xi_1 = t_1 + t_1^{-1}$, then $\xi_b = t_1^b + t_1^{-b}$, and $t_1^{\frac{s+1}{2}} = 1$. We have the following solutions:

$$f = s - 1, \quad \chi_l = -(s-1), \quad \chi_\mu = \chi_\nu = -1, \quad \chi_m = \chi_n = 1,$$
$$\chi_\alpha = 0 \text{ if } \epsilon_\alpha = 1, \quad \chi_\beta = -(t_1^b + t_1^{-b}) \text{ if } \epsilon_\beta = -1.$$

We find that $x = 0$ is satisfied by all these solutions; and that $y = -(1+\epsilon)$ and $z = 1 + \epsilon$ are satisfied by all except $t_1 = -1$, which can happen only when $\epsilon = 1$. These solutions furnish $\frac{1}{4}(s - \epsilon)$ characters.

The total number of characters thus obtained is

$$4 + 2 + \tfrac{1}{4}(s - 4 - \epsilon) + \tfrac{1}{4}(s - 2 + \epsilon) + \tfrac{1}{4}(s - 2 + \epsilon) + \tfrac{1}{4}(s - \epsilon) = s + 4.$$

which is equal to the number of classes of conjugate substitutions.

Finally, we readily find that for all these $s+4$ characters the second proportionality factor e is equal to f, where e is defined by

$$\frac{hf}{e}=\sum_{\sigma} h_{\sigma}\,\chi_{\sigma}\,\chi_{\sigma'}.$$

Below is given a table of the group-characters, N denoting the number of characters in the respective columns.

N	1	1	2	2	$\frac{s-\varepsilon-4}{4}$	$\frac{s+\varepsilon-2}{4}$	$\frac{s+\varepsilon-2}{4}$	$\frac{s-\varepsilon}{4}$
χ_λ	1	s	$\frac{s+\varepsilon}{2}$	$\frac{s-\varepsilon}{2}$	$s+1$	$s+1$	$s-1$	$s-1$
χ_l	1	s	$\frac{s+\varepsilon}{2}$	$-\frac{s-\varepsilon}{2}$	$s+1$	$-(s+1)$	$s-1$	$-(s-1)$
χ_μ	1	0	$\frac{\varepsilon\pm\sqrt{\varepsilon s}}{2}$	$\frac{-\varepsilon\pm\sqrt{\varepsilon s}}{2}$	1	1	-1	-1
χ_ν	1	0	$\frac{\varepsilon\mp\sqrt{\varepsilon s}}{2}$	$\frac{-\varepsilon\mp\sqrt{\varepsilon s}}{2}$	1	1	-1	-1
χ_m	1	0	$\frac{\varepsilon\pm\sqrt{\varepsilon s}}{2}$	$\frac{\varepsilon\mp\sqrt{\varepsilon s}}{2}$	1	-1	-1	1
χ_n	1	0	$\frac{\varepsilon\mp\sqrt{\varepsilon s}}{2}$	$\frac{\varepsilon\pm\sqrt{\varepsilon s}}{2}$	1	-1	-1	1
$\chi(R^a)$	1	1	$\frac{(1+\varepsilon)(-1)^a}{2}$	$\frac{(1-\varepsilon)(-1)^a}{2}$	r^a+r^{-a}	$r_1^a+r_1^{-a}$	0	0
$\chi(S^b)$	1	-1	$-\frac{(1-\varepsilon)(-1)^b}{2}$	$-\frac{(1+\varepsilon)(-1)^b}{2}$	0	0	$-t^b-t^{-b}$	$-t_1^b-t_1^{-b}$

where r, r_1, t, t_1, are the roots (except ± 1) of the respective equations $r^{\frac{s-1}{2}}=1$, $r_1^{\frac{s-1}{2}}=-1$, $t^{\frac{s+1}{2}}=1$, $t_1^{\frac{s+1}{2}}=-1$.

§ 6.

By the use of the following theorems, due to Frobenius, we are able to deduce the group-characters of F from those of H.

*If G is an invariant subgroup of the group H then every character of $\frac{H}{G}$ is also a character of H.**

* Über die Darstellung der endlichen Gruppen durch lineare Substitutionen, Berliner Sitzungsberichte, 1897, p. 995.

*In order that a character of H may belong to the group $\frac{H}{G}$ it is necessary and sufficient that it have the same value for all elements of G. Then it has also equal values for every two elements of H which are equivalent mod. G.**

In the present case the invariant subgroup G of H is composed of the substitutions $(l) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and (λ) the identity. Then those and only those characters of H for which $\chi_l = \chi_\lambda$ belong to the group $\frac{H}{G} = F$; and since every character of F belongs to H, we obtain in this way all the characters of F. The classes (μ) and (m), also (ν) and (n), are equivalent mod. G. Hence we can write down at once the table of characters for F.

N	1	1	2	$\frac{s-\varepsilon-4}{4}$	$\frac{s+\varepsilon-2}{4}$
χ_λ	1	s	$\frac{s+\varepsilon}{2}$	$s+1$	$s-1$
χ_μ	1	0	$\frac{\varepsilon \pm \sqrt{\varepsilon s}}{2}$	1	-1
χ_ν	1	0	$\frac{\varepsilon \mp \sqrt{\varepsilon s}}{2}$	1	-1
$\chi(R^a)$	1	1	$\frac{(1+\varepsilon)(-1)^a}{2}$	$r^a + r^{-a}$	0
$\chi(S^b)$	1	-1	$\frac{(1-\varepsilon)(-1)^b}{2}$	0	$-t^b - t^{-b}$

where r and t are the roots (except ± 1) of the respective equations $r^{\frac{s-1}{2}} = 1$, $t^{\frac{s+1}{2}} = 1$.

II.

The Binary Linear Homogeneous Group H_1 in the $GF[2^n]$.

The order of H_1 is $h = 2^n(2^{2n} - 1)$, and the determinant of each substitution is unity. The group is holoedrically isomorphic with the group of all binary linear fractional substitutions in the $GF[2^n]$.

* Über Relationen zwischen den Charakteren einer Gruppe und denen ihrer Untergruppen, Berliner Sitzungsberichte, 1898, p. 510.

We define $x = \alpha + \delta$ as the invariant of the substitution $\begin{pmatrix}\alpha, & \beta\\ \gamma, & \delta\end{pmatrix}$.

The substitution $R = \begin{pmatrix}\rho, & 0\\ 0, & \rho^{-1}\end{pmatrix}$, where ρ is a primitive root of the $GF[2^n]$, generates a cyclic group of order $s-1$. R^a is conjugate to R^{-a} and is always distinct from it. Hence the powers of R represent $\frac{s-2}{2}$ classes, each containing $s(s+1)$ substitutions.

Let σ be a primitive root of $\sigma^{s+1} = 1$. The substitution $S = \begin{pmatrix}\sigma, & 0\\ 0, & \sigma^{-1}\end{pmatrix}$ is of period $s+1$. S^b is conjugate to S^{-b} and is distinct from it; thus the powers of S represent $\frac{s}{2}$ classes, each containing $s(s-1)$ substitutions.

The substitution $T = \begin{pmatrix}1, & 1\\ 0, & 1\end{pmatrix}$ of period two and invariant zero is one of $s^2 - 1$ conjugate substitutions. We denote this class by (0) and the identity by (λ).

The total number of classes of conjugate substitutions is $s+1$.

Below is given the table of group-characters.

N	1	1	$2^{n-1}-1$	2^{n-1}
χ_λ	1	2^n	2^n+1	2^n-1
χ_0	1	0	1	-1
$\chi(R^a)$	1	1	$r^a + r^{-a}$	0
$\chi(S^b)$	1	-1	0	$-t^b - t^{-b}$

where r and t are the roots (except unity) of the respective equations $r^{s-1} = 1$, $t^{s+1} = 1$. As before $e = f$.

III.

The Binary Linear Fractional Group F_1 in the $GF[p^n]$, $p > 2$, of all Determinants not Zero.

The order of F_1 is $h = s(s^2 - 1)$. The substitutions will be supposed written in the normal form, *i. e.*, of determinant unity or a particular not-square in the $GF[p^n]$.

We shall denote the determinant $\alpha\delta - \beta\gamma$ of the substitution $V = \begin{pmatrix}\alpha, & \beta\\ \gamma, & \delta\end{pmatrix}$ by τ, where $\tau = 1$, or ν a particular not-square; and we shall call $\pm \frac{1}{2}(\alpha + \delta)$ the invariant of V.

If two substitutions have not the same determinant they are not conjugate. If two substitutions (neither the identity) have the same determinant and the same invariant, they are conjugate under F_1.

By canonical form theory we find that all the substitutions of the group can be reduced to one or another of the following canonical forms:

A) $$R = \begin{pmatrix} \rho, & 0 \\ 0, & \rho^{-1}\tau \end{pmatrix}, \ \rho \text{ a mark} \neq 0 \text{ of the } GF[p^n];$$

B) $$S = \begin{pmatrix} \sigma, & 0 \\ 0, & \sigma^{-1}\tau \end{pmatrix}, \ \sigma \text{ a mark} \neq 0 \text{ of the } GF[p^{2n}];$$

C) $$T = \begin{pmatrix} 1, & 1 \\ 0, & 1 \end{pmatrix},$$

where σ satisfies a quadratic equation belonging to and irreducible in the $GF[p^n]$.

A) The substitution
$$R = \begin{pmatrix} \rho, & 0 \\ 0, & \rho^{-1}\nu \end{pmatrix}$$
where $\rho^2\nu^{-1}$ is a primitive root of the $GF[p^n]$, is of period $s-1$. With the exception of $R^{\frac{s-1}{2}}$ which is conjugate only to itself, R^a is conjugate to R^{-a} and is distinct from it. We have therefore $\frac{s-1}{2}$ classes represented by the powers of R, each containing $s(s+1)$ substitutions, except $R^{\frac{s-1}{2}}$, the class represented by which contains $\frac{1}{2}s(s+1)$ substitutions.

B) The group of all binary linear fractional substitutions in the $GF[p^n]$ of determinant $\neq 0$ is holoedrically isomorphic with the group* of binary hyperorthogonal substitutions in the $GF[p^{2n}]$ of determinant a mark of the $GF[p^n]$ when taken fractionally, viz.,
$$U = \begin{pmatrix} A, & B \\ -\bar{B}, & \bar{A} \end{pmatrix} \qquad (A\bar{A} + B\bar{B} = \pi),$$
where $\bar{A} \equiv A^s$ is the conjugate of A with respect to the $GF[p^n]$, and π is a mark $\neq 0$ of the $GF[p^n]$.

Consider the substitution
$$S = \begin{pmatrix} \sigma, & 0 \\ 0, & \sigma^{-1}\nu \end{pmatrix},$$
where σ is a primitive root of the equation $\sigma^{s+1} = \nu$. Since ν is an *arbitrary* not-square we may suppose that it is a primitive root of the $GF[p^n]$. Then σ is a

* Dickson, Linear Groups, § 144, Cor.

primitive root of the $GF[p^{2n}]$, and consequently S is of period $s+1$. With the exception of $S^{\frac{s+1}{2}}$ which is conjugate only to itself, S^b is conjugate to S^{-b} and is distinct from it. We have therefore $\frac{s+1}{2}$ classes represented by the powers of S, each containing $s(s-1)$ substitutions, except $S^{\frac{s+1}{2}}$, the class represented by which contains $\frac{1}{2}s(s-1)$ substitutions.

The classes represented by the powers of $R(S)$ are characterized by the property that $x^2-\tau$ is a square (not-square) in the $GF[p^n]$, where $\tau=\nu$ or 1 according as the index is odd or even.

The substitution

$$T_\mu = \binom{1,\ \mu}{0,\ 1},\quad \mu \text{ a mark} \neq 0 \text{ of the } GF[p^n],$$

of invariant ± 1 and determinant unity, is one of s^2-1 conjugate substitutions forming a class (μ).

The total number of classes of conjugate substitutions is $s+2$.

Below is given the table of group-characters.

N	1	1	1	1	$\frac{s-3}{2}$	$\frac{s-1}{2}$
χ_λ	1	1	s	s	$s+1$	$s-1$
χ_μ	1	1	0	0	1	-1
$\chi(R^{2a})$	1	1	1	1	$r^{2a}+r^{-2a}$	0
$\chi(S^{2b})$	1	1	-1	-1	0	$-t^{2b}-t^{-2b}$
$\chi(R^{2a+1})$	1	-1	1	-1	$r^{2a+1}+r^{-(2a+1)}$	0
$\chi(S^{2b+1})$	1	-1	-1	1	0	$-t^{2b+1}-t^{-(2b+1)}$

where r and t are the roots (except ± 1) of $r^{s-1}=1$ and $t^{s+1}=1$ respectively. As before $e=f$.

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VITA.

I was born near Lemonville, Ont., Can., and received my elementary education at Lemonville Public School and at Markham High School. In 1900 I received the degree of A. B., and in 1901 the degree of A. M., from McMaster University, Toronto, Ont. The following three years, with the exception of one quarter, I spent in the University of Chicago, in the departments of Mathematics and Astronomy, holding a fellowship in Mathematics 1901–1904.

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